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MATHEMATICS

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MAXIMUM LIKELIHOOD ESTIMATION OF ORDERED
PROBABILITIES ¹⁾

BY

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1. Introduction

The problem treated in this paper concerns the maximum likelihood estimation of partially or completely ordered probabilities.

Consider k independent series of independent trials, each trial resulting in a success or a failure. The i -th series consists of n_i trials with a_i ²⁾ successes and $b_i = n_i - a_i$ failures; π_i is the (unknown) probability of a success for each trial of the i -th series ($i = 1, 2, \dots, k$) and $\pi_1, \pi_2, \dots, \pi_k$ satisfy the inequalities

$$(1.1) \quad \alpha_{i,j} (\pi_i - \pi_j) \leq 0 \quad (i, j = 1, 2, \dots, k),$$

where

$$(1.2) \quad \begin{cases} 1. & \alpha_{i,i} = -\alpha_{i,i}, \\ 2. & \alpha_{i,j} = 0 \text{ for } m_0 \text{ pairs of values } (i, j) \text{ with } i < j, \\ 3. & \alpha_{i,j} = 1 \text{ for } m_1 \text{ pairs of values } (i, j) \text{ with } i < j, \end{cases}$$

$$(1.3) \quad \begin{cases} m_0 + m_1 = \binom{k}{2}, \\ m_1 \geq 1 \end{cases}$$

and, if $i < h < j$ then

$$(1.4) \quad \alpha_{i,j} = 1 \text{ if } \alpha_{i,h} = \alpha_{h,j} = 1 \text{ (transitivity).}$$

In section 2 and 3 methods will be described by means of which the maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$ may be found, i.e. the values of $x_1, x_2, \dots, x_k$ which maximize

$$(1.5) \quad L = L(x_1, x_2, \dots, x_k) \stackrel{\text{def}}{=} \sum_{i=1}^k \{a_i \lg x_i + b_i \lg (1 - x_i)\}$$

in the domain

$$(1.6) \quad D: \begin{cases} \alpha_{i,j} (x_i - x_j) \leq 0, \\ 0 \leq x_i \leq 1 \end{cases} \quad (i, j = 1, 2, \dots, k).$$

¹⁾ Report SP 50 of the Statistical Department of the Mathematical Centre.

²⁾ Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by printing their symbols in bold type.

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Unless explicitly stated otherwise L will only be considered in this domain D ; the maximum likelihood estimates will throughout this paper be denoted by $p_1, p_2, \dots, p_k$, the point in D where L assumes its maximum.

Let further

$$(1.7) \quad L_i = L_i(x_i) \stackrel{\text{def}}{=} a_i \lg x_i + b_i \lg (1-x_i) \quad (i = 1, 2, \dots, k)$$

and

$$(1.8) \quad f_i = \frac{a_i}{n_i} \quad (i = 1, 2, \dots, k).$$

Further the restrictions $\pi_i \leq \pi_j$ (i.e. $\alpha_{i,j} = 1$) satisfying

$$(1.9) \quad \alpha_{i,h} \alpha_{h,j} = 0 \text{ for each } h \text{ between } i \text{ and } j$$

will be denoted by $R_1, R_2, \dots, R_s$. Each R_λ thus corresponds with one pair (i, j) , i.e. R_λ denotes the relation $\pi_i \leq \pi_j$, satisfying (1.9); this pair (i, j) will be denoted by (i_λ, j_λ) . Because of the transitivity relations (1.4) the system $R_1, R_2, \dots, R_s$ is equivalent to (1.1).

In section 4 some examples will be given.

Remarks:

1. Every set of restrictions $R_1, R_2, \dots, R_s$ represents a convex domain.
2. It will be clear that partially or completely ordered probabilities can always be numbered in such a way that they satisfy (1.1).
3. If $m_0 = 0$ then (1.1) is equivalent to

$$(1.10) \quad \pi_1 \leq \pi_2 \leq \dots \leq \pi_k.$$

This special case has been solved independently by MIRIAM AYER, H. D. BRUNK, G. M. EWING, W. T. REID, EDWARD SILVERMAN [2] and the present author [1].

2. The maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$

In this section the following theorem will be proved.

Theorem I: L possesses a unique maximum and if $p'_1, p'_2, \dots, p'_k$ are the maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$ under the restrictions $R_1, \dots, R_{\lambda-1}, R_{\lambda+1}, \dots, R_s$ then the maximum likelihood estimates $p_1, p_2, \dots, p_k$ under the restrictions $R_1, R_2, \dots, R_s$ satisfy the relations

$$(2.1) \quad \begin{cases} 1. & p_i = p'_i \quad (i = 1, 2, \dots, k) \text{ if } p'_{i_\lambda} \leq p'_{j_\lambda}, \\ 2. & p_{i_\lambda} = p_{j_\lambda} \text{ if } p'_{i_\lambda} > p'_{j_\lambda}. \end{cases}$$

Proof: The R_λ have not been arranged in a special order. Therefore we may, without any loss of generality, take $\lambda = s$.

The uniqueness of the maximum of L will be proved by induction. If $s = 0$ then it is well known that L possesses for any k a unique maximum and

$$(2.2) \quad p_i = f_i \quad (i = 1, 2, \dots, k).$$

Further $s = 0$ if $k = 1$; therefore L possesses a unique maximum and (2.2)

holds for $k=1$. Now suppose that it has been proved that L possesses a unique maximum for the following two cases:

$$(2.3) \quad \begin{cases} 1. & k \text{ series of trials with } s-1 \text{ restrictions} \\ 2. & k-1 \text{ series of trials with } s-1 \text{ or less restrictions,} \end{cases}$$

where $k \geq 2$, $s \geq 1$ and consider a case with k series of trials and s restrictions. Then it follows from (2.3.1) that L possesses a unique maximum under the restrictions $R_1, R_2, \dots, R_{s-1}$ i.e. there exists exactly one point $p'_1, p'_2, \dots, p'_k$ satisfying these restrictions and maximizing L .

Now the following two cases may be distinguished.

1. $p'_i \leq p'_j$; then $p'_1, p'_2, \dots, p'_k$ satisfy all restrictions $R_1, R_2, \dots, R_s$. Therefore in this case L possesses a unique maximum under the restrictions $R_1, R_2, \dots, R_s$ and

$$(2.4) \quad p_i = p'_i \quad (i = 1, 2, \dots, k).$$

2. $p'_i > p'_j$; then (2.1.2) may be proved as follows. Consider a fixed point $x_1, x_2, \dots, x_k$ in D with $x_i < x'_i$. It will be clear that such points exist. Then if

$$(2.5) \quad \begin{cases} X_i(\beta) \stackrel{\text{def}}{=} (1-\beta)x_i + \beta p'_i & (i = 1, 2, \dots, k), \\ 0 \leq \beta \leq 1, \end{cases}$$

we have

$$(2.6) \quad X_i(0) = x_i, \quad X_i(1) = p'_i \quad (i = 1, 2, \dots, k)$$

and for each β with $0 \leq \beta \leq 1$, $X_1(\beta), X_2(\beta), \dots, X_k(\beta)$ is a point satisfying the restrictions $R_1, R_2, \dots, R_{s-1}$, but not necessarily R_s (cf. remark 1 in the foregoing section). Therefore if

$$(2.7) \quad \beta_0 \stackrel{\text{def}}{=} \frac{x_{j_s} - x_{i_s}}{x_{j_s} - x_{i_s} + p'_{i_s} - p'_{j_s}}$$

then

$$(2.8) \quad \begin{cases} 1. & 0 < \beta_0 < 1, \\ 2. & X_{i_s}(\beta_0) = X_{j_s}(\beta_0), \end{cases}$$

i.e. $X_1(\beta_0), X_2(\beta_0), \dots, X_k(\beta_0)$ is a point satisfying the restrictions

$$R_1, R_2, \dots, R_s,$$

i.e. a point in D .

Further $L\{X_1(\beta), X_2(\beta), \dots, X_k(\beta)\}$ is for fixed values of $x_1, x_2, \dots, x_k$ a function of β , say $g(\beta)$, and from (1.5) and (2.5) it follows that

$$(2.9) \quad \frac{d^2 g(\beta)}{d\beta^2} = \sum_{i=1}^k (p'_i - x_i)^2 \frac{-n_i \{X_i(\beta)\}^2 + 2a_i X_i(\beta) - a_i}{\{X_i(\beta)\}^2 \{1 - X_i(\beta)\}^2}.$$

Further we have

$$(2.10) \quad -n_i \{X_i(\beta)\}^2 + 2a_i X_i(\beta) - a_i \begin{cases} < 0 \text{ for all } \beta \text{ if } 0 < a_i < n_i, \\ = -n_i \{X_i(\beta)\}^2 \text{ if } a_i = 0, \\ = -n_i \{1 - X_i(\beta)\}^2 \text{ if } a_i = n_i \end{cases}$$

and from (2.9) and (2.10) it follows, $p'_i - x_i$ being $\neq 0$ for at least one value of i (viz. for $i=i_s$ or/and $i=j_s$), that

$$(2.11) \quad \frac{d^2 g(\beta)}{d\beta^2} < 0.$$

Further we have

$$(2.12) \quad L\{p'_1, p'_2, \dots, p'_k\} > L\{X_1(\beta), X_2(\beta), \dots, X_k(\beta)\}$$

for each β with $0 \leq \beta < 1$ and from (2.6), (2.11) and (2.12) it follows that $g(\beta)$ is an increasing function of β in the interval $0 \leq \beta \leq 1$.

Thus for each point $x_1, x_2, \dots, x_k$ in D with $x_{i_s} < x_{j_s}$, D contains a point $X_1, X_2, \dots, X_k$ with

$$(2.13) \quad \begin{cases} 1. & X_{i_s} = X_{j_s}, \\ 2. & L(X_1, X_2, \dots, X_k) > L(x_1, x_2, \dots, x_k), \end{cases}$$

i.e. $L(x_1, x_2, \dots, x_k)$ attains its maximum under the restrictions $R_1, R_2, \dots, R_s$ for $x_{i_s} = x_{j_s}$. Substituting this into (1.5) the two terms with $i=i_s$ and $i=j_s$ are reduced to one term of the form

$$(a_{i_s} + a_{j_s}) \lg x_{i_s} + (b_{i_s} + b_{j_s}) \lg (1 - x_{i_s}).$$

This means that the two series of trials in question are to be pooled. The uniqueness of the maximum of L under these restrictions then follows from (2.3.2).

By applying theorem I repeatedly and using the well known solution of the problem for the case that $s=0$ a solution may be obtained. This may, however, lead to a rather complicated procedure, which can often be simplified by applying the special theorems mentioned in the following section.

3. Some special theorems

Theorem II: If $\alpha_{i,j}(f_i - f_j) \leq 0$ for each pair of values (i, j) then

$$(3.1) \quad p_i = f_i \quad (i = 1, 2, \dots, k).$$

Proof: This follows immediately from the fact that in this case the maximum of L in D coincides with the maximum of L in the domain: $0 \leq x_i \leq 1$ ($i = 1, 2, \dots, k$). The theorem also follows from theorem I.

The following theorem will be immediately clear.

Theorem III: If $i_1, i_2, \dots, i_v$ is a set of values satisfying

$$(3.2) \quad \alpha_{i_1, i_1} = \alpha_{i_1, i_2} = \dots = \alpha_{i_1, i_v} = 0 \quad \text{for each } i \neq i_1, i_2, \dots, i_v$$

then the maximum likelihood estimates of $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_v}$ are those values of $x_{i_1}, x_{i_2}, \dots, x_{i_v}$ which maximize $L_{i_1} + L_{i_2} + \dots + L_{i_v}$ in the domain

$$(3.3) \quad D': \begin{cases} \alpha_{i_h, i_{h'}}(x_{i_h} - x_{i_{h'}}) \leq 0, \\ 0 \leq x_{i_h} \leq 1 \end{cases} \quad (h, h' = 1, 2, \dots, v).$$

For the proof of the theorems IV and V we need the following lemma.

Lemma I: If $x_1, x_2, \dots, x_k$ is any point in D such that for some pair of values (i, j)

$$(3.4) \quad \begin{cases} 1. & x_i < x_j, \\ 2. & f_i \geq f_j \\ 3. & \alpha_{h,i} \geq \alpha_{h,j} \quad \text{for each } h < j, \\ 4. & \alpha_{i,h} \leq \alpha_{j,h} \quad \text{for each } h > i, \end{cases}$$

then a number x exists satisfying

$$(3.5) \quad \begin{cases} 1. & x_1, x_2, \dots, x_k \text{ is also a point in } D \text{ if } x \text{ is substituted for } \\ & x_i \text{ and } x_j, \\ 2. & L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j). \end{cases}$$

Proof:

The following cases may be distinguished

$$(3.6) \quad x_i < x_j \leq f_i; \text{ then take } x = x_j,$$

$$(3.7) \quad f_i \leq x_i < x_j; \text{ then take } x = x_i,$$

$$(3.8) \quad x_i < f_i < x_j; \text{ then take } x = f_i.$$

It may easily be proved that this number x satisfies (3.5.2). For (3.6) e.g. we have

$$(3.9) \quad L_j(x) = L_j(x_j)$$

and

$$(3.10) \quad x_i < x \leq f_i.$$

From (3.10) follows

$$(3.11) \quad L_i(x) > L_i(x_i)$$

and (3.5.2) follows from (3.9) and (3.11).

For the cases (3.7) and (3.8) it may be proved in a similar way by means of (3.4.2) that L satisfies (3.5.2).

In order to prove that this number x satisfies (3.5.1) it is sufficient to prove that

$$(3.12) \quad \begin{cases} 1. & \alpha_{h,i}(x_h - x) \leq 0 \quad \text{for each } h < i, \\ 2. & \alpha_{i,h}(x - x_h) \leq 0 \quad \text{for each } h > i, \\ 3. & \alpha_{h,j}(x_h - x) \leq 0 \quad \text{for each } h < j, \\ 4. & \alpha_{j,h}(x - x_h) \leq 0 \quad \text{for each } h > j. \end{cases}$$

From the fact that x satisfies

$$(3.13) \quad x_i \leq x \leq x_j$$

and the fact that $x_1, x_2, \dots, x_k$ is a point in D it follows that

$$(3.14) \quad \begin{cases} 1. & \alpha_{h,i}(x_h - x) \leq \alpha_{h,i}(x_h - x_i) \leq 0 \quad \text{for each } h < i, \\ 2. & \alpha_{i,h}(x - x_h) \leq \alpha_{i,h}(x_j - x_h) \leq 0 \quad \text{for each } h > j. \end{cases}$$

Further it follows from (3.4.3) and (3.4.4) that

$$(3.15) \quad \begin{cases} 1. & \alpha_{i,h}(x-x_h) = \alpha_{j,h}(x-x_h) \text{ for each } h > i \text{ with } \alpha_{i,h} = 1, \\ 2. & \alpha_{h,j}(x_h-x) = \alpha_{h,i}(x_h-x) \text{ for each } h < j \text{ with } \alpha_{h,j} = 1 \end{cases}$$

and (3.12) follows from (3.14) and (3.15).

Theorem IV: *If for some pair of values (i, j) with $i < j$*

$$(3.16) \quad \alpha_{i,j}(f_i - f_j) > 0$$

and

$$(3.17) \quad \begin{cases} 1. & \alpha_{i,h} = \alpha_{h,j} = 0 \text{ for each } h \text{ between } i \text{ and } j, \\ 2. & \alpha_{h,i} = \alpha_{h,j} \text{ for each } h < i, \\ 3. & \alpha_{i,h} = \alpha_{j,h} \text{ for each } h > j, \end{cases}$$

then

$$(3.18) \quad p_i = p_j.$$

Proof: From (3.16) and (3.17) it follows that

$$(3.19) \quad \begin{cases} 1. & f_i > f_j, \\ 2. & \alpha_{h,i} = \alpha_{h,j} \text{ for each } h < j, \\ 3. & \alpha_{i,h} = \alpha_{j,h} \text{ for each } h > i. \end{cases}$$

Now suppose that $x_1, x_2, \dots, x_k$ is a point in D with

$$(3.20) \quad x_i < x_j.$$

From lemma I, (3.19) and (3.20) it follows then that a number x exists such that $x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{j-1}, x, x_{j+1}, \dots, x_k$ is a point in D and

$$(3.21) \quad L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j).$$

Thus for each point $x_1, x_2, \dots, x_k$ in D with $x_i < x_j$ a point $x'_1, x'_2, \dots, x'_k$ in D exists with

$$(3.22) \quad \begin{cases} 1. & x'_i = x'_j, \\ 2. & L(x'_1, x'_2, \dots, x'_k) > L(x_1, x_2, \dots, x_k), \end{cases}$$

i.e. L attains its maximum for $x_i = x_j$ and (3.18) then follows from the uniqueness of this maximum.

Remarks:

4. This theorem is also related to theorem I. If R_λ represents the restriction $\pi_i \leq \pi_j$ it follows from (3.22) that L attains its maximum under the restrictions $R_1, \dots, R_{\lambda-1}, R_{\lambda+1}, \dots, R_s$ for $x_i \geq x_j$, giving $p'_i \geq p'_j$; from (2.1) then follows: $p_i = p_j$.

5. If $m_0 = 0$, i.e. if the probabilities $\pi_1, \pi_2, \dots, \pi_k$ satisfy the inequalities

$$(3.23) \quad \pi_1 \leq \pi_2 \leq \dots \leq \pi_k,$$

then each pair of values (i, j) with $j = i + 1$ satisfies (3.17). Therefore we have in this case

$$(3.24) \quad p_i = p_{i+1} \text{ for each } i \text{ with } f_i > f_{i+1}.$$

From theorem IV it follows that if there is a pair of values (i, j) satisfying (3.16) and (3.17) then the problem may be reduced to the case of $k-1$ series of trials with $s-1$ (or less) restrictions by substituting $x_i = x_j$ into $L(x_1, x_2, \dots, x_k)$, i.e. by pooling the i -th and j -th series of trials.

Theorem V: *If (i, j) is a pair of values satisfying*

$$(3.25) \quad f_i \leq f_j$$

and

$$(3.26) \quad \begin{cases} 1. & \alpha_{i,j} = 0, \\ 2. & \alpha_{h,i} \leq \alpha_{h,j} \text{ for each } h < i, \\ 3. & \alpha_{i,h} \geq \alpha_{j,h} \text{ for each } h > j, \end{cases}$$

then

$$(3.27) \quad p_i \leq p_j.$$

Proof: Suppose $x_1, x_2, \dots, x_k$ is a point in D with

$$(3.28) \quad x_i > x_j.$$

From lemma I, (3.25), (3.26) and (3.28) it follows then in the same way as in theorem IV that for each point $x_1, x_2, \dots, x_k$ in D with $x_i > x_j$ a point $x'_1, x'_2, \dots, x'_k$ in D exists with

$$(3.29) \quad \begin{cases} 1. & x'_i = x'_j, \\ 2. & L(x'_1, x'_2, \dots, x'_k) > L(x_1, x_2, \dots, x_k). \end{cases}$$

Now D also contains points with $x_i < x_j$ and therefore L attains its maximum for $x_i \leq x_j$; (3.27) then follows from the uniqueness of the maximum.

By means of theorem V a new restriction may be introduced. This is sometimes useful as may be seen from example 2 of section 4.

Theorem VI: *If (i, j) is a pair of values with*

$$(3.30) \quad \alpha_{i,j} = 0,$$

if D' is the sub-domain of D where $x_i \leq x_j$ and if $p'_1, p'_2, \dots, p'_k$ is the point where L assumes its maximum in D' then

$$(3.31) \quad \begin{cases} 1. & p_1 = p'_1, p_2 = p'_2, \dots, p_k = p'_k \text{ if } p'_i < p'_j, \\ 2. & p_i \geq p_j \text{ if } p'_i = p'_j. \end{cases}$$

Proof:

First consider the case that $p'_i < p'_j$. For this case it may be proved that $p_i < p_j$, by showing that $p_i \geq p_j$ leads to a contradiction. Because if $p_i \geq p_j$ then it may be proved in the same way as in theorem I that L attains its maximum in D' for $x_i = x_j$, i.e. then we have $p'_i = p'_j$. Thus $p_i < p_j$ if $p'_i < p'_j$ and from the uniqueness of the maximum of L then follows

$$(3.32) \quad p_1 = p'_1, p_2 = p'_2, \dots, p_k = p'_k \text{ if } p'_i < p'_j.$$

Now consider the case that $p'_i = p'_j$, i.e. the case that L attains its maximum in D' for $x_i = x_j$; then L attains its maximum in D for $x_i \geq x_j$, i.e. $p_i \geq p_j$.

Thus if the maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$ in D' are known then the problem is solved by means of theorem VI if $p'_i < p'_j$ (cf. (3.31.1)) or the new restriction $\pi_j \leq \pi_i$ may be introduced if $p'_i = p'_j$ (cf. (3.31.2)).

Remark:

6. Theorem V is related to theorem VI. Taking in theorem V for D' the subdomain of D where $x_j \leq x_i$, it follows from (3.29) that $p'_j = p'_i$. From (3.31.2) then follows: $p_i \leq p_j$.

In general it is much more simple to apply the theorems II–V than I or VI. In some situations however II–V are not applicable and then I or VI have to be used. This will be illustrated in the examples of the following section.

4. Examples

Example 1 (complete ordering; theorems II and IV):

Suppose $k=4$, $m_0=0$ ($\pi_1 \leq \pi_2 \leq \pi_3 \leq \pi_4$) and

$$(4.1) \quad \left\{ \begin{array}{ccccc} i & 1 & 2 & 3 & 4 \\ a_i & 4 & 3 & 10 & 8 \\ n_i & 10 & 5 & 30 & 15 \\ f_i & 0,4 & 0,6 & 0,33 & 0,53. \end{array} \right.$$

From (4.1) and (3.24) it follows that

$$(4.2) \quad p_2 = p_3$$

and the problem is reduced to the case of $k-1=3$ series of trials by pooling the second and third series of trials:

$$(4.3) \quad \left\{ \begin{array}{cccc} i & 1 & 2(+3) & 4 \\ a'_i & 4 & 13 & 8 \\ n'_i & 10 & 35 & 15 \\ f'_i & 0,4 & 0,37 & 0,53. \end{array} \right.$$

From (4.3) and (3.24) it then follows that

$$(4.4) \quad p_1 = p_2$$

and the problem is reduced to the case of $k-2=2$ series of trials with

$$(4.5) \quad \left\{ \begin{array}{ccc} i & 1(+2+3) & 4 \\ a''_i & 17 & 8 \\ n''_i & 45 & 15 \\ f''_i & 0,38 & 0,53. \end{array} \right.$$

From (4.5), (4.2), (4.4) and theorem II then follows

$$(4.6) \quad p_1 = p_2 = p_3 = 0,38, \quad p_4 = 0,53.$$

Example 2 (incomplete ordering; theorems II, IV and V):

Suppose $k=5$, $m_1=6$, $m_0=4$

$$(4.7) \quad \begin{cases} i & 1 & 2 & 3 & 4 & 5 \\ a_i & 7 & 13 & 15 & 2 & 12 \\ n_i & 10 & 20 & 30 & 5 & 15 \\ f_i & 0,7 & 0,65 & 0,5 & 0,4 & 0,8 \end{cases}$$

and

$$(4.8) \quad \alpha_{1,2} = \alpha_{1,3} = \alpha_{2,4} = \alpha_{3,5} = 1.$$

Then the pair of values $i=2$, $j=4$ satisfies (3.16) and (3.17). Therefore we have

$$(4.9) \quad p_2 = p_4$$

and the problem is reduced to the case of $k-1=4$ series of trials with $m'_1=4$, $m'_0=2$,

$$(4.10) \quad \begin{cases} i & 1 & 2(+4) & 3 & 5 \\ a'_i & 7 & 15 & 15 & 12 \\ n'_i & 10 & 25 & 30 & 15 \\ f'_i & 0,7 & 0,6 & 0,5 & 0,8 \end{cases}$$

and

$$(4.11) \quad \alpha'_{1,2} = \alpha'_{1,3} = \alpha'_{3,5} = 1.$$

For these 4 series of trials the pair $i=3$, $j=2$ and the pair $i=2$, $j=5$ satisfy (3.25) and (3.26). From theorem V then follows that L attains its maximum for

$$(4.12) \quad x_1 \leq x_3 \leq x_2 \leq x_5$$

and from (4.9), (4.10) and (4.12) follows

$$(4.13) \quad p_1 = p_3 = 0,55, \quad p_2 = p_4 = 0,6, \quad p_5 = 0,8.$$

Example 3 (incomplete ordering; theorem I or VI):

Suppose $k=4$, $m_0=m_1=3$,

$$(4.14) \quad \begin{cases} i & 1 & 2 & 3 & 4 \\ a_i & 7 & 18 & 13 & 10 \\ n_i & 10 & 30 & 20 & 25 \\ f_i & 0,7 & 0,6 & 0,65 & 0,4 \end{cases}$$

and

$$(4.15) \quad \alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = 1.$$

For this case the theorems II–V cannot be applied and we use therefore theorem I.

Take $i_1=1$ and $j_1=1$ (i.e. omit the restriction $\pi_1 \leq \pi_4$), then p'_1, p'_2, p'_3, p'_4 are those values of x_1, x_2, x_3, x_4 which maximize L in the domain

$$(4.16) \quad \begin{cases} x_1 \leq x_2, & x_3 \leq x_4, \\ 0 \leq x_i \leq 1 & (i = 1, 2, 3, 4). \end{cases}$$

From theorem III and IV then follows

$$(4.17) \quad p'_1 = p'_2 = 0,63, \quad p'_3 = p'_4 = 0,51$$

and from theorem I and (4.17) (cf. (2.1.2))

$$(4.18) \quad p_1 = p_4.$$

In this way the problem is reduced to the case of $k-1=3$ series of trials with

$$(4.19) \quad \begin{cases} i & 3 & 1(+4) & 2 \\ a'_i & 13 & 17 & 18 \\ n'_i & 20 & 35 & 30 \\ f'_i & 0,65 & 0,49 & 0,6 \end{cases}$$

and

$$(4.20) \quad \alpha'_{3,1} = \alpha'_{1,2} = 1.$$

From (4.18), (4.19) and (4.20) follows

$$(4.21) \quad p_1 = p_3 = p_4 = 0,55, \quad p_2 = 0,6.$$

This problem may also be solved by means of theorem VI as follows. If we take $i=2, j=3$ then D' is the domain

$$(4.22) \quad \begin{cases} x_1 \leq x_2 \leq x_3 \leq x_4, \\ 0 \leq x_i \leq 1 & (i = 1, 2, 3, 4). \end{cases}$$

The estimates p'_1, p'_2, p'_3, p'_4 then follow from (3.24):

$$(4.23) \quad p'_1 = p'_2 = p'_3 = p'_4 = 0,56.$$

From theorem VI and (4.23) then follows (cf. (3.31.2))

$$(4.24) \quad p_2 \geq p_3.$$

Introducing the restriction $\pi_3 \leq \pi_2$ the problem is reduced to the case of 4 series of trials with (4.14) and

$$(4.25) \quad \alpha'_{1,2} = \alpha'_{1,4} = \alpha'_{3,2} = \alpha'_{3,4} = 1.$$

Then the pair $i=3, j=1$ and the pair $i=4, j=2$ satisfy (3.25) and (3.26). From theorem V then follows that L attains its maximum for

$$(4.26) \quad x_3 \leq x_1 \leq x_4 \leq x_2$$

and from (4.14), (4.26) and (3.24) then follows

$$(4.27) \quad p_1 = p_3 = p_4 = 0,55, \quad p_2 = 0,6.$$

Example 4 (incomplete ordering; theorem VI):

Suppose $k=8$, $m_0=13$, $m_1=15$,

$$(4.22) \quad \begin{cases} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a_i & 8 & 22 & 13 & 25 & 20 & 21 & 32 & 2 \\ n_i & 10 & 40 & 20 & 50 & 30 & 50 & 50 & 5 \\ f_i & 0,8 & 0,55 & 0,65 & 0,5 & 0,67 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4.23) \quad \alpha_{1,2}'' = \alpha_{1,4}'' = \alpha_{3,4}'' = \alpha_{4,5}'' = \alpha_{5,6}'' = \alpha_{5,8}'' = \alpha_{7,8}'' = 1.$$

For this case the theorems II–V cannot be applied and we use therefore theorem VI.

Taking for (i, j) the pair $i=6$, $j=7$ the estimates $p'_1, p'_2, \dots, p'_8$ are the values of $x_1, x_2, \dots, x_8$ which maximize L in the domain

$$(4.24) \quad D': \begin{cases} x_1 \leq x_2, x_1 \leq x_4, x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8, \\ 0 \leq x_i \leq 1. \quad (i=1, 2, \dots, 8). \end{cases}$$

These estimates may be found as follows. From theorem IV follows

$$(4.25) \quad p'_5 = p'_6, p'_7 = p'_8.$$

Thus the problem of finding $p'_1, p'_2, \dots, p'_8$ is reduced to the case of $k-2=6$ series of trials with $m'_0=5$, $m'_1=10$,

$$(4.26) \quad \begin{cases} i & 1 & 2 & 3 & 4 & 5(+6) & 7(+8) \\ a'_i & 8 & 22 & 13 & 25 & 41 & 34 \\ n'_i & 10 & 40 & 20 & 50 & 80 & 55 \\ f'_i & 0,8 & 0,55 & 0,65 & 0,5 & 0,51 & 0,62 \end{cases}$$

and

$$(4.27) \quad \alpha'_{1,2} = \alpha'_{1,4} = \alpha'_{3,4} = \alpha'_{4,5} = \alpha'_{5,7} = 1.$$

To these 6 series of trials we apply theorem I, taking for R_1 the restriction $\pi_1 \leq \pi_4$. Then we find by means of the theorems II, III and IV in a similar way as in example 3

$$(4.28) \quad p'_1 = p'_4.$$

In this way the problem of finding $p'_1, p'_2, \dots, p'_8$ is reduced to the case of $k-3=5$ series with $m''_0=2$, $m''_1=8$,

$$(4.29) \quad \begin{cases} i & 3 & 1(+4) & 2 & 5(+6) & 7(+8) \\ a''_i & 13 & 33 & 22 & 41 & 34 \\ n''_i & 20 & 60 & 40 & 80 & 55 \\ f''_i & 0,65 & 0,55 & 0,55 & 0,51 & 0,62 \end{cases}$$

and

$$(4.30) \quad \alpha''_{3,1} = \alpha''_{1,2} = \alpha''_{1,5} = \alpha''_{5,7} = 1.$$

From theorem IV then follows

$$(4.31) \quad p'_1 = p'_3$$

and the problem is reduced to the case of $k-4=4$ series of trials with $m''_0=2$, $m''_1=4$

$$(4.32) \quad \begin{pmatrix} i & 1(+3+4) & 2 & 5(+6) & 7(+8) \\ \alpha''_i & 46 & 22 & 41 & 34 \\ n''_i & 80 & 40 & 80 & 55 \\ f''_i & 0,58 & 0,55 & 0,51 & 0,62 \end{pmatrix}$$

and

$$(4.33) \quad \alpha'''_{1,2} = \alpha'''_{1,5} = \alpha'''_{1,7} = 1.$$

From theorem V then follows

$$(4.34) \quad p'_2 \leq p'_7 \text{ and } p'_5 \leq p'_2$$

and from (3.24), (4.25), (4.28) and (4.31)

$$(4.35) \quad p'_1 = p'_3 = p'_4 = p'_5 = p'_6 = 0,54, p'_2 = 0,55, p'_7 = p'_8 = 0,62.$$

From theorem VI and (4.35) it follows, p'_6 being smaller than p'_7 (cf. (3.31.1))

$$(4.36) \quad p_i = p'_i \quad (i = 1, 2, \dots, 8).$$

The method used in this example for finding the estimates $p_1, p_2, \dots, p_8$ is not the only possible way. The problem may also be solved by exclusively applying theorem I, taking any of the restrictions $R_1, R_2, \dots, R_8$ for R_i or by exclusively applying theorem VI, taking for i and j any of the pairs of values (i, j) with $\alpha_{i,j}=0$.

Remark:

7. The indicated procedure may be generalized to several cases of ordered parameters of other probability distributions, e.g. parameters of Poisson distributions and means of normally distributed variables with known variances. This generalization and the properties of the estimates are being investigated.

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